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European Journal of Combinatorics 27 (2006) 436–443
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Group connectivity of graphs with diameter at most 2

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Received 29 January 2004; accepted 16 November 2004
Available online 25 March 2005

Abstract

Let G be an undirected graph, A be an (additive) abelian group and $A^* = A - \{0\}$. A graph G is A-connected if G has an orientation $D(G)$ such that for every function $b : V(G) \mapsto A$ satisfying $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \mapsto A^*$ such that at each vertex $v \in V(G)$, the amount of f values on the edges directed out from v minus the amount of f values on the edges directed into v equals $b(v)$. In this paper, we investigate, for a 2-edge-connected graph G with diameter at most 2, the group connectivity number $\Lambda_g(G) = \min\{n : G$ is A-connected for every abelian group A with $|A| \geq n\}$, and show that any such graph G satisfies $\Lambda_g(G) \leq 6$. Furthermore, we show that if G is such a 2-edge-connected diameter 2 graph, then $\Lambda_g(G) = 6$ if and only if G is the 5-cycle; and when G is not the 5-cycle, then $\Lambda_g(G) = 5$ if and only if G is the Petersen graph or G belongs to two infinite families of well characterized graphs.

1. Introduction

Graphs in this paper are finite and may have loops and multiple edges. Undefined terms and notation are from [1]. Throughout the paper, $\mathbf{Z}_n$ denotes the cyclic group of order n, for some integer $n \geq 2$.

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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2004.11.001

Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let $\mathrm{tail}(e) = u$ and $\mathrm{head}(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^+(v) = \{e \in E(D) : v = \mathrm{tail}(e)\}, \quad \text{and} \quad E_D^-(v) = \{e \in E(D) : v = \mathrm{head}(e)\}.$$

Let A denote an (additive) abelian group with identity 0, and let $F(G, A)$ denote the set of all functions from $E(G)$ to A, and $F^*(G, A)$ the set of all functions from $E(G)$ to $A - \{0\}$. Given a function $f \in F(G, A)$, let $\partial f : V(G) \mapsto A$ be given by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A function $b : V(G) \mapsto A$ is called an *A-valued zero sum function on G* if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero sum functions on G is denoted by $Z(G, A)$. Given $b \in Z(G, A)$, if G has an orientation D and a function $f \in F^*(G, A)$ such that $\partial f = b$, then f is an *(A, b)-nowhere-zero flow* (abbreviated as an (A, b)-NZF) under the orientation. A graph G is *A-connected* if for any b, G has an (A, b)-NZF. For an abelian group A, let $\langle A \rangle$ be the family of graphs that are A-connected. It is observed in [5] that the property $G \in \langle A \rangle$ is independent of the orientation of G, and that every graph in $\langle A \rangle$ is 2-edge-connected.

For convenience, we also use 0 to denote the zero value constant function in $F(G, A)$. Then an *A-nowhere-zero flow* (abbreviated as an A-NZF) in G is an $(A, 0)$-NZF. For a positive integer k, when $A = \mathbf{Z}$, the additive group of all the integers, a function $f \in F^*(G, \mathbf{Z})$ is a nowhere-zero k-flow (abbreviated as a k-NZF) of G if $|f(e)| < k$, $\forall e \in E(G)$. Tutte ([14], also [4]) showed that a graph G has an A-NZF if and only if G has an $|A|$-NZF.

The concept of A-connectivity was introduced by Jaeger et al. in [5], where A-NZF's were successfully generalized to A-connectivity. A similar concept to the group connectivity was independently introduced in [7], with a different motivation from [5]. For more on the literature of the nowhere-zero-flow problems, see [14,4] or [15].

For a 2-edge-connected graph G, define the *flow number* of G as

$$\Lambda(G) = \min\{k : G \text{ has a } k\text{-NZF}\}$$

and the *group connectivity number* of G as

$$\Lambda_g(G) = \min\{k : \text{ if } A \text{ is an abelian group with } |A| \geq k, \text{ then } G \in \langle A \rangle\}.$$

Tutte conjectured (see [4]) that if a 2-edge-connected graph G does not have a subgraph contractible to P_{10}, the Petersen graph, then $\Lambda(G) \leq 4$. This conjecture has been proved for cubic graphs [12], and the general case remains open.

The main purpose of this note is to show that for 2-edge-connected graphs with diameter at most 2, the group connectivity number is at most 6 with the 5-cycle C_5 being the only such graph with $\Lambda_g(C_5) = 6$. Moreover, we will also show that if G is a 2-edge-connected graph with diameter at most 2 such that $G \neq C_5$, then $\Lambda_g(G) \leq 5$. All such graphs whose group connectivity number is exactly 5 will be characterized.

2. Preliminaries

In this section, we present some of the known results which will be utilized in our proofs.

For a subset $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge e in X and deleting resulting loops. Note that even G is a simple graph, the contraction G/X may have multiple edges. For convenience, we define $G/\emptyset = G$, and write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then we write G/H for $G/E(H)$.

Proposition 2.1 (*Proposition 3.2 of [9]*). *Let A be an abelian group with $|A| \geq 3$. Then $\langle A \rangle$ satisfies each of the following:*

(C1) $K_1 \in \langle A \rangle$,
(C2) *if $G \in \langle A \rangle$ and $e \in E(G)$, then $G/e \in \langle A \rangle$,*
(C3) *if H is a subgraph of G and if both $H \in \langle A \rangle$ and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Lemma 2.2 (*[5] and [9]*). *Let C_n denote a cycle of n vertices. Then $C_n \in \langle A \rangle$ if and only if $|A| \geq n + 1$. (Equivalently, $\Lambda_g(C_n) = n + 1$.)*

Lemma 2.3 (*[5]*). *Let G be a connected graph with n vertices and m edges. Then $\Lambda_g(G) = 2$ if and only if $n = 1$ (and so G has m loops with a single vertex).*

Let $O(G) = \{\text{odd degree vertices of } G\}$. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph Γ_R such that $O(\Gamma_R) = R$. As in [1], $\kappa'(G)$ denotes the edge-connectivity of a graph G.

Theorem 2.4 (*[2]*). *Suppose that G is one edge short of having two edge-disjoint spanning trees. Then G is collapsible if and only if $\kappa'(G) \geq 2$.*

Lemma 2.5 (*[8]*). *Let G be a collapsible graph and let A be an abelian group with $|A| = 4$. Then $G \in \langle A \rangle$.*

Lemma 2.6 (*[10]*). *Let A be an abelian group with $|A| \geq 3$, and let G be a connected graph. If for each $e \in E(G)$, G has a subgraph $H_e \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Theorem 2.7 (*[3]*). *Let $m > n > 0$ be integers. If A is an abelian group with $|A| = 4$, then $K_{2,n} \notin \langle A \rangle$. Furthermore,*

$$\Lambda_g(K_{m,n}) = \begin{cases} 5 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 3 & \text{if } n \geq 4. \end{cases}$$

3. Graphs with diameter at most 2

Let G be a connected graph. For any pair of vertices $u, v \in V(G)$, let $d(u, v)$ denote the length of a shortest (u, v)-path in G. Then the *diameter* of G equals $\max\{d(u, v) : u, v \in V(G)\}$.

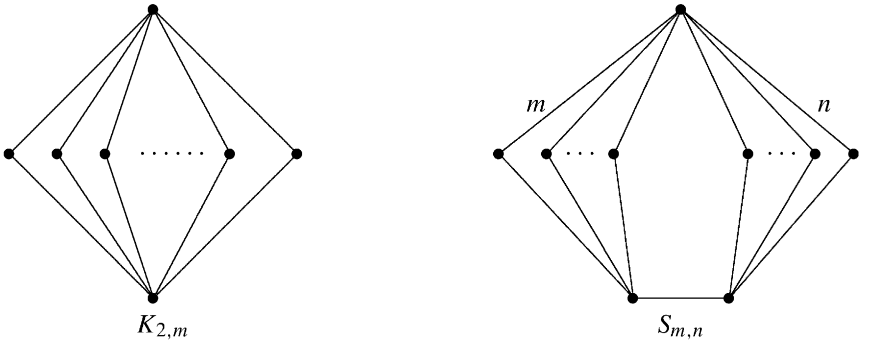

Fig. 1. $K_{2,m}$ and $S_{m,n}$.

In 1989, it was shown [6] that if G is a 2-edge-connected graph with diameter at most 2, then $\Lambda(G) \leq 5$, where equality holds if and only if $G \cong P_{10}$. In this section, we shall consider what would happen if we replaced Λ by Λ_g in this result. Lemma 2.2 indicates that $\Lambda_g(C_5) = 6$ and Theorem 2.7 suggests that P_{10} is not the only diameter 2 graph with group connectivity number equaling 5.

Let m, n be two positive integers and let $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,n}$. Let u_1, u_2 denote the two nonadjacent vertices of degree m in H_1; v_1, v_2 denote the two nonadjacent vertices of degree n in H_2. Then $S_{m,n}$ is the graph obtained from the disjoint union of H_1 and H_2 by identifying u_1 with v_1 and by adding a new edge e joining u_2 to v_2 (see Fig. 1). Note that $S_{1,1} \cong C_5$, the 5-cycle.

The main results of the paper are the following.

Theorem 3.1. *If G is a 2-edge-connected loopless graph with diameter at most 2, then $\Lambda_g(G) \leq 6$, where $\Lambda_g(G) = 6$ if and only if $G = C_5$, the 5-cycle.*

Theorem 3.2. *If G is a 2-edge-connected loopless graph with diameter at most 2 such that $G \neq C_5$, then $\Lambda_g(G) \leq 5$. Moreover, when G is one such graph, the following are equivalent:*

(i) $\Lambda_g(G) = 5$.
(ii) *$G \cong P_{10}$, the Petersen graph, or for some integers m, n with $m > n > 0$, $G \cong S_{m,n}$, or G has a collapsible subgraph H with $|V(H)| \leq 2$ such that $G/H \cong K_{2,m}$ for some integer $m > 1$.*

A few lemmas and former results are needed for the proofs of Theorems 3.1 and 3.2. Recall that a graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph Γ_R such that $O(\Gamma_R) = R$. For a graph G, let G' denote the graph obtained from G by successively contracting all maximal collapsible subgraphs of G. Then G' is called the *reduction* of G. Catlin [2] showed that the reduction of any graph exists and is unique.

Lemma 3.3 (*Theorem 3 of [6]*). *Let G be a 2-edge-connected graph with diameter at most 2, and let G' denote the reduction of G. Then exactly one of the following must hold:*

(i) *G is collapsible,*
(ii) $G \cong P_{10}$,
(iii) $G' \cong C_5$,
(iv) $G' \cong K_{2,m}$ *for some integer* $m \geq 2$,
(v) $G' \cong S_{m,n}$ *for some integer* m, n *with* $m > n > 0$.

Theorem 3.4 ([5]). *If G has two edge-disjoint spanning trees, then for any abelian group A with $|A| \geq 4$, $G \in \langle A \rangle$.*

Lemma 3.5. *Let G be a loopless graph. If one of the following holds:*

(i) $G \cong P_{10}$, *or*
(ii) *G has a collapsible subgraph H with $|V(H)| \leq 2$ such that $G/H \cong K_{2,m}$ for some $m \geq 2$, or*
(iii) $G \cong S_{m,n}$ *for some* m, n *with* $m > n > 0$,

then $\Lambda_g(G) = 5$.

Proof. We first show that if (ii) holds, then $\Lambda_g(G) = 5$. By Theorem 2.7, $\Lambda_g(K_{2,m}) = 5$. Suppose that G has a subgraph H with $|V(H)| \leq 2$ such that $G/H \cong K_{2,m}$. Then $\Lambda_g(G) \geq \Lambda_g(G/H) = \Lambda_g(K_{2,m}) = 5$.

If $|V(H)| = 1$, then $\Lambda_g(G) = \Lambda_g(K_{2,m}) = 5$. Therefore, we assume that $|V(H)| = 2$. Since H is collapsible and since $|V(H)| = 2$, H must have two edge-disjoint spanning trees. Let A be an abelian group A with $|A| \geq 5$. By Theorem 3.4, $H \in \langle A \rangle$. Since $G/H \cong K_{2,m} \in \langle A \rangle$, by (C3) of Proposition 2.1, $G \in \langle A \rangle$, and so $\Lambda_g(G) = 5$.

Next, we do the same if (iii) holds. Suppose that some $S_{m,n} \in \langle A \rangle$. $S_{m,n}$ can be contracted to a $K_{2,m+n}$, and so by (C2) of Proposition 2.1, $K_{2,m+n} \in \langle A \rangle$ as well. Thus $\Lambda_g(S_{m,n}) \geq \Lambda_g(K_{2,m+n}) = 5$. Since $m > n > 0$, $S_{m,n}$ must contain a subgraph $H \cong K_{2,m}$ with $m \geq 2$. By Theorem 2.7, $\Lambda_g(H) = 5$. Every edge in $S_{m,n}/H$ lies in a cycle of length at most 4, and so by Lemmas 2.2 and 2.6, $\Lambda_g(S_{m,n}/H) \leq 5$. Since $\Lambda_g(H) = 5$ and since $\Lambda_g(S_{m,n}/H) \leq 5$, it follows by (C3) of Proposition 2.1 that $\Lambda_g(S_{m,n}) \leq 5$, and so we conclude that $\Lambda_g(S_{m,n}) = 5$.

It remains to show that $\Lambda_g(P_{10}) = 5$. It is well known that $\Lambda(P_{10}) = 5$, and so we have $\Lambda_g(P_{10}) \geq \Lambda(P_{10}) = 5$. To see that $\Lambda_g(P_{10}) = 5$, it suffices to show that for any abelian group A with $|A| \geq 5$, $P_{10} \in \langle A \rangle$. Since every edge of P_{10} lies in a 5-cycle, by Lemmas 2.2 and 2.6, if $|A| \geq 6$, then $P_{10} \in \langle A \rangle$. It remains to show that $P_{10} \in \langle \mathbf{Z}_5 \rangle$.

First, we shall show that if e, e' are incident with the same vertex $v \in V(P_{10})$, then $H = (P_{10} - e)/\{e'\} \in \langle \mathbf{Z}_5 \rangle$. This is because that H contains a 4-cycle L such that in H/L, every edge lies in a cycle of length at most 4. Therefore, by Lemmas 2.2 and 2.6, $H/L \in \langle \mathbf{Z}_5 \rangle$. By Lemma 2.2, $L \in \langle \mathbf{Z}_5 \rangle$, and so by (C3) of Proposition 2.1, $H \in \langle \mathbf{Z}_5 \rangle$.

Now we can prove that $P_{10} \in \langle \mathbf{Z}_5 \rangle$. Let $b \in Z(P_{10}, \mathbf{Z}_5)$. If $b \equiv 0$, then since $\Lambda(P_{10}) = 5$, P_{10} has a $(\mathbf{Z}_5, 0)$-NZF. Therefore, we assume that for some vertex $v \in V(P_{10})$, $b(v) \neq 0 \in \mathbf{Z}_5$. Let e be an edge joining v to another vertex u in P_{10}, and let e, e', e'' be the three edges incident with v in P_{10}. Then define $H = (P_{10} - e)/\{e'\}$. By the definition of contraction, we may assume that $V(H) = V(P_{10}) - \{v\}$.

Define $b' : V(H) \mapsto \mathbf{Z}_5$ by

$$b'(z) = \begin{cases} b(u) + b(v) & \text{if } z = u \\ b(z) & \text{if } z \neq u. \end{cases}$$

Then $b' \in Z(H, \mathbf{Z}_5)$. Since $H \in \langle \mathbf{Z}_5 \rangle$, under a certain orientation of H, there exists $f' \in F^*(H, \mathbf{Z}_5)$ such that $\partial f' = b'$.

By the definition of a contraction, we can view $E(H) = E(P_{10}) - \{e, e'\}$, and so this orientation of H can be viewed as an orientation of the edges of $P_{10} - \{e, e'\}$ in P_{10}. Without loss of generality, we may assume that in this orientation of P_{10}, e'' is directed into v. We extend this orientation to an orientation of P_{10} by orienting both e and e' from v.

Extend f' to a function $f \in F^*(P_{10}, \mathbf{Z}_5)$ as follows: $\forall_z \in E(P_{10})$,

$$f(z) = \begin{cases} f'(z) & \text{if } z \in E(P_{10}) - \{e, e'\} \\ f(e'') & \text{if } z = e' \\ b(v) & \text{if } z = e. \end{cases}$$

Then $f \in F^*(P_{10}, \mathbf{Z}_5)$ such that $\partial f = b$. Therefore, $P_{10} \in \langle \mathbf{Z}_5 \rangle$, as desired. □

Lemma 3.6. *Let A be an abelian group with $|A| = 4$, and let G be a 2-edge-connected graph of diameter at most 2. Then $G \notin \langle A \rangle$ if and only if either $G \cong C_5$ or* (ii) *of Theorem* 3.2 *holds.*

Proof. Suppose first that $G \in \langle A \rangle$. Then by (C2) of Proposition 2.1 and by Lemma 3.5, G cannot be contractible to a $K_{2,m}$ or to P_{10}. In particular, $G \ncong S_{m,n}$. Thus we cannot have $G \cong C_5$ and neither can (ii) of Theorem 3.2 hold.

We apply Lemma 3.3 to show the converse. Assume that $G \notin \langle A \rangle$. By Lemma 2.5, (i) of Lemma 3.3 does not hold. As (ii) of Lemma 3.3 implies (ii) of Theorem 3.2, we may assume that (i) and (ii) of Lemma 3.3 do not hold either. Hence one of (iii), (iv) or (v) of Lemma 3.3 must hold. If G', the reduction of G, is isomorphic to an $S_{m,n}$ for some positive integers m and n (including $S_{1,1} = C_5$), and if $G \neq G'$, then G has some maximal nontrivial subgraphs $H_1, H_2, \ldots, H_k$ such that $G/(H_1 \cup H_2 \cup \cdots \cup H_k) \cong S_{m,n}$. For each i, since H_i is nontrivial, H_i must have at least two vertices and so the diameter of G would exceed 2, contrary to the assumption that the diameter of G is at most 2. Therefore, if (iii) or (v) of Lemma 3.3 holds, then $G \cong S_{m,n}$ for some positive integers m and n, and so (ii) of Theorem 3.2 must hold.

Finally, we assume that $G' \cong K_{2,m}$ for some $m \geq 2$. Then either $G \cong K_{2,m}$ and we are done, or G has maximal nontrivial collapsible subgraphs $H_1, H_2, \ldots, H_k$ such that $G/(H_1 \cup H_2 \cup \cdots \cup H_k) \cong K_{2,m}$. Since the diameter of G is at most 2, we must have $k = 1$ with $|V(H_1)| \leq 2$, and so (ii) of Theorem 3.2 holds. This completes the proof of Lemma 3.6. □

Theorem 3.7 ([11]). *Let $\delta(G)$ denote the minimum degree of a graph G. If $\delta(G) \geq 4$, then G has a nontrivial subgraph which has two edge-disjoint spanning trees.*

Theorem 3.8 ([13]). *Let $d > 0$ be an integer. If G has diameter d and if every cycle of G has length at least $2d + 1$, then G must be a regular graph.*

Lemma 3.9. *Let G be a 2-edge-connected graph with diameter at most 2. Suppose that G does not have a cycle of length at most 4. If G does not have a nontrivial subgraph which has two edge-disjoint spanning trees, then G is isomorphic to C_5 or to P_{10}.*

Proof. By Theorem 3.7, we have $2 \leq \delta(G) \leq 3$. By Theorem 3.8, G must be either 2-regular or 3-regular, and so, in these cases, G is isomorphic to either C_5 or to P_{10} (see page 82 of [6] for a detailed proof). □

Proof of Theorem 3.2. Let A be an abelian group of order at least 4. Suppose first that $|A| \geq 5$. We argue by induction on $|V(G)|$ to show that $G \in \langle A \rangle$. By Lemma 2.2, any 2-edge-connected graph with at most four vertices must be in $\langle A \rangle$. Therefore, we assume that $|V(G)| \geq 5$. If G has a nontrivial subgraph H which has two edge-disjoint spanning trees, or which is isomorphic to a cycle of length at most 4, then by Theorem 3.4 or Lemma 2.2, $H \in \langle A \rangle$. As the diameter will not increase under contraction, G/H is also 2-edge-connected with diameter at most 2. By induction, $G/H \in \langle A \rangle$ and so by (C3) of Proposition 2.1, $G \in \langle A \rangle$.

Therefore, we assume that G satisfies the hypotheses of Lemma 3.9, and so $G \in \{C_5, P_{10}\}$. By the assumption of Theorem 3.2, $G \neq C_5$, and so we must have $G \cong P_{10}$. By Lemma 3.5, $G \in \langle A \rangle$. This proves that $\Lambda_g(G) \leq 5$.

Suppose that $\Lambda_g(G) = 5$. If G is collapsible, then by Lemma 2.5, we would have $\Lambda_g(G) \leq 4$, contrary to the assumption that $\Lambda_g(G) = 5$. Therefore, G cannot be collapsible. Since G cannot be C_5, (ii), (iv) or (v) of Lemma 3.3 must hold and so (i) of Theorem 3.2 implies (ii) of Theorem 3.2. Conversely, Lemma 3.5 shows that (ii) of Theorem 3.2 implies (i) of Theorem 3.2. □

Proof of Theorem 3.1. Since G is 2-edge-connected with diameter at most 2, every edge lies in a cycle of length at most 5, and so by Lemmas 2.2 and 2.6, $\Lambda_g(G) \leq 6$. If $G \neq C_5$, then by Theorem 3.2, $\Lambda_g(G) \leq 5$. On the other hand, Lemma 2.2 shows that $\Lambda_g(C_5) = 6$. This completes the proof. □

Acknowledgements

The authors would like to thank the referees for their helpful suggestions that improved the presentation of this paper.

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